

A graphical approach to the teaching of introductory calculus

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This research project investigated the use of computer software (ANUgraph) in the development of calculus concepts. One group of students completed the calculus unit following a traditional teacher-exposition method. The other group covered the same content using a graphical approach, which relied heavily upon the use of the computer software to introduce and reinforce key concepts. Results indicate that the lessons proved successful in increasing students' understanding of the fundamental concepts of calculus.

Rationale and Theoretical Framework

A traditional method of teaching Differential Calculus is to introduce students to a formula which includes a limit. A typical lesson sequence often involves students completing "first principles" exercises before learning a rule for differentiation and this sometimes leads to a separation of the process of differentiation from the concept of the gradient function or derivative. Tall & Vinner (1981) identified the concept image / concept definition conflicts that occur in students when using the traditional approach and student's difficulties with the algebraic manipulations involved in the "first principles" definition were described by White (1992) and Arnold (1994). White (1991) developed a new approach to the introduction of calculus concepts which attempts to avoid the conceptual difficulties described above and uses practical contexts in which to investigate the nature of change. This work coupled with the developments that have occurred in the technological field as described by Tall (1986) and in particular the widespread availability of "Mathematical Function Plotters" have lead to the development of the approach used in this study.

The emphasis, in this study, is on the use of the computer to model the concepts being introduced from a graphical perspective. Thus the lesson contexts are generally graphical rather than practical, in the sense used by White (1992). Particularly significant, is the use of the graphical approach developed by Tall (1986) to introduce the concept of the derivative using the "zoom" function available in the software. This allows for the conceptual problems associated with limits and function notation to be circumvented thus enabling the students to gain a clear concept definition of the gradient of a function prior to having to engage in the algebraic manipulation associated with the derivative. Also of particular importance here is the approach taken to develop the elementary rules of differentiation from the graphs of functions rather than from the "first principles definition".

The Study

Methodology

A quasi-experimental design was used to test the effects of the use of a Mathematical Function Plotter (ANUgraph) and associated pedagogical approaches in the teaching of Elementary Calculus to a Year 11 - 2 Unit Mathematics Class. A Trial group was used to assist in the development and testing of pre-test and post-test materials and lesson plans. The selected treatment group (a class of 16 students) and the control group (a class of 23 students) were tested using the pre-test and it was found that there was no

significant statistical difference between the groups on the overall test results. [$t(39) = 0.618$ with $p=0.470$, (> 0.05)]. The treatment using the graphical computer-based approach was applied and the control group was taught using a traditional “limit based first principles” approach. Both groups were tested using the Post Test. Results revealed a 12% difference in overall means, ($p= 0.035$; $p < 0.05$ $n=39$)

Treatment

Having developed and fine tuned all materials the Pre-Test was administered to both the treatment and control groups. This was followed by the lesson sequence and Post-Tests. The responses of students in *pre-tests* and *post-tests* were used to ascertain their perceptions and understanding of key concepts associated with introductory calculus. These concepts included the nature and definition of function, the definition of derivative, the use of derivative in relation to properties of functions, the use of second derivative as a tool in investigating the nature of functions.

Teaching Strategies

Example 1: Comparative Sequence of Stages in the Teaching Process

Stage	Treatment Group	Control Group
1	Review of Functions & Function Notation	Introduction to Limits
2	Gradient of a Curve and the Gradient Function using ANUGraph Students Derive Formula : If $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$	Complete algebraic exercises on limits.
3	Differentiation of Sums and Differences	Introduction to gradient of a curve & derivation & use of the formula $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ Exercises on the use of formula
4	Derivative as a Rate of Change and Increasing/Decreasing Functions.	Rules for Differentiation Formula :If $f(x) = x^n$ then $f'(x) = n \cdot x^{n-1}$ <i>Conditions for Differentiability</i>
5	Definition of the Derivative as a Limit	Differentiation of Sums and Differences
6	Curves that Have Some Interesting Points (<i>Conditions for Differentiability</i>)	The Chain Rule- Exercises
7	The Chain Rule	Further Exercises on Chain Rule
8	The Product Rule/Quotient Rule	The Product Rule and Quotient Rule- Exercises
9	Consolidation and Practice	Exercises on differentiation
10	The First Derivative and Curve Sketching	Turning Points and Local Maxima and Minima- Exercises
11	The Second Derivative and Curve Sketching	The Second derivative and curve sketching - Exercises
12	Further Applications , consolidation and practice.	Consolidation and practice.

Major Findings

Comparisons Between Groups

Pre-Test Results: Initially the mean and standard deviation for each key concept category was calculated for the Pre-Test providing an overall summary of the results. In addition to this, the results were analysed to ascertain if any significant statistical difference occurred between the performance of each group on each concept/skill area as well as the overall test result. An unpaired two-tailed *t* - test was used for this purpose and the results for each question and the total test score are include in the Table below.

Group A - Treatment Group; Group B – Control Group; *t* -values are unpaired 2-tailed, with 39 df and level of significance at 95%. Scores are percentages

	Group	Mean	SD	<i>t</i> (39)	<i>p</i>	Comments
TOTAL	A	52	15	0.73	0.470	No sig. diff. on overall total.
SCORES	B	49	17			

The results indicate that the overall result of $t(39) = 0.618$ with $p=0.470$, (> 0.05) supports the null hypothesis, leading to the conclusion that there is no significant difference between the groups on the overall scores.

Post-Test Data

Post-Test results for the Treatment and control groups were also analysed in a similar manner. The tables below show the results for each of the groups on each question.

	Group	Mean	SD	<i>t</i> (39)	<i>p</i>	Comments
TOTAL	A	59	20	2.187	0.035	A significant difference occurs on the overall test results. The difference of 12% between means is statistically sig., @ 95% confidence limits. $t(39) = 2.178$ with $p = 0.035$, (<0.05 .)
SCORES	B	47	19			

Conclusions & Implications

The results obtained demonstrate that the students in the treatment group were able to successfully complete the standard algorithms of differentiation and interpret these results in a superior manner to those in the control group. In comparing the two approaches taken here one of the elementary differences is that the treatment group spent more time on “playing” with the graphs of the functions and less time on practice exercises in the traditional sense. It would appear therefore that the students’ competence in the algorithms of differentiation did not suffer, but was in fact higher, and was therefore enhanced by this approach. Perhaps the reason for this was related to the fact that the graphical approach places the concepts in a context thus enhancing the

students links between concept image and concept definition. The use of the “magnification” principle in determining the gradient at a point and its initial link to the rate of change at that point, avoids the complications of the limiting process and associated notations. The better performance of the treatment group on the standard algorithms of differentiation appears to be unrelated to prior understanding of the limiting process. The fact that there was little difference in the initial understanding of the limiting process as measured in the pre-test substantiates this. The results in the post-test also supports the view that the graphical approach taken in this study enhances the concept image that the students have in relation to the derivative and its application.

The implications for teaching programs and curricula are that, (1) students should be given access to this type of technology in the context of the Mathematics curriculum in order to dynamically explore the graphical approach to the teaching of calculus; (2) the work supports earlier work by others in the development of a context-based approach to the teaching of calculus; and (3) a change in the emphasis in senior curricular which emphasise the graphical approach rather in addition to the more traditional algebraic limiting value approach is worthy of consideration.

Appendix 1.

A Sample Lesson Outline (Stage 2)

Introduction to the Concept of the Gradient Function (Stage 2)

The concept of speed being equivalent to the slope of a graph is developed from a known situation (ie. the Travel Graph).

The concept of average speed is developed and the link to the Chord between the two points on the curve is established.

The “zoom in” feature of the software is then used to investigate the curve in the neighbourhood of $x = 1$ to establish the gradient at this point.

This gradient is then linked to the concept of “Instantaneous Speed” which is then linked to the Gradient of the Tangent.

Later Exercises are used to develop the concept of the Gradient Function and hence the Derivative, using the “Differentiate Option” on ANUgraph to display a table of values of the Gradient Functions for various functions.

These values are plotted to produce graphs from which the students “guess” the function rules for the Gradient Function.

After several different repetitions of this process the rule for determining the derivative of a polynomial type function is derived from the pattern.

Stage 2 (Details of Teaching Strategies)

Student Investigation: (Teacher directed).

A vehicle travels from A to B and its distance D kilometres from the starting point A is given by the function $D = t^2$; where t is the time taken in hours.

- How far is it from the origin after 5 hours?
- How long does it take to travel 16 km?
- What is its average speed over; a) the first 2 hours? (b) the first 3 hours? (c) the first 4 hours? (d) between $t = 2$ hrs and $t = 5$ hrs.
- What would its speed be when $t = 1$ hour? Explain how you got your answer.

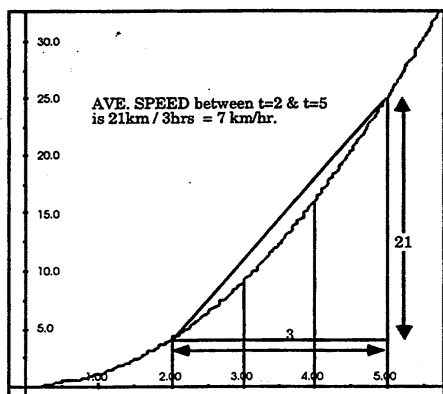
1. Using ANUGraph

Using ANUgraph, the students then plot the graph of $D = t^2$

(Using $F(x)$ for D and the x axis for t) (ie. Graph $F(x) = x^2$)

Ensuring that the "Show Coordinates" option is selected (Special Menu), so that coordinate values on the graph can be identified using the mouse pointer, students investigate the answers to the preceding questions with reference to the graph.. Also it is pointed out that since we are investigating the position of the vehicle after it leaves its starting position we need only consider $t > 0$.

Speed: The teacher now makes the following observations with the group;



The average speed of the vehicle is given by

$$\text{Speed} = \frac{\text{Distance Travelled}}{\text{Time Taken}}$$

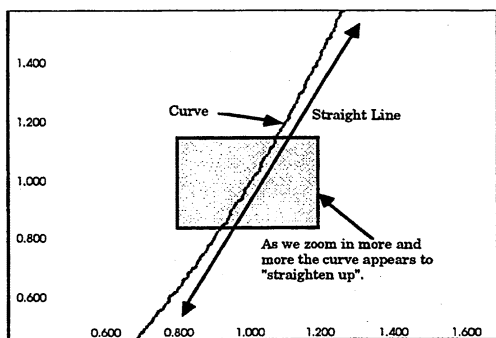
Average speed can be determined by joining two points on the curve and calculating

$$\text{Ave. Speed} = \frac{\text{Vertical Rise}}{\text{Horizontal Run}}$$

for the two points chosen. (See diagram). Therefore, on our graph *speed* is equivalent to the

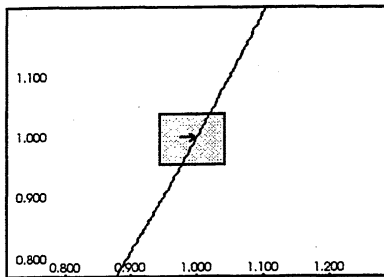
$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

The teacher then proceeds to make the necessary connections to the instantaneous speed as follows;



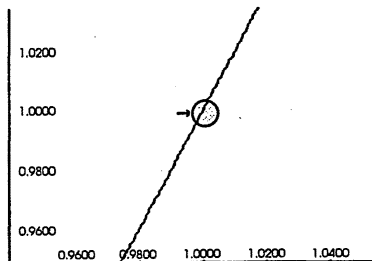
We can see therefore that the *Slope* or *Gradient* of the function can be used to determine the *speed*, however there is a problem here in that we are dealing with a curve rather than a straight line.

The instantaneous speed is a little more difficult to find but not impossible. Finding the speed when $t = 1$ hr requires an examination of the graph in the "neighbourhood" of $t = 1$.

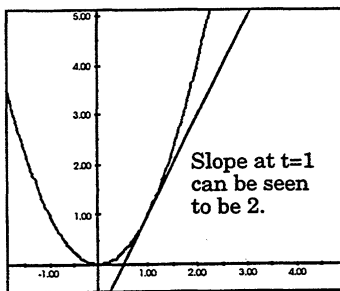


Using the “Zoom” feature on ANUgraph we can “zoom in” on the point (1, 1) and examine the slope to establish the speed of the vehicle at this point.

We notice that as the zoom area gets smaller the graph appears to get straighter.



We can therefore calculate using the formula - Slope = $\frac{\text{Rise}}{\text{Run}}$ that the slope of the graph near $t = 1$ would be about 2. (As we zoom in closer and closer we could draw a line with slope exactly 2, just touching the graph at (1, 1). (ie a tangent to the curve at $t=1$). In our problem this then gives us the “Instantaneous Speed” when $t = 1$ (ie. at time $t = 1$ hour, the vehicle is travelling at 2 km/hr.)

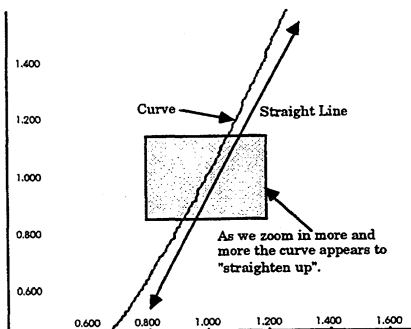


To assist us with the process of calculating instantaneous speed, ANUgraph will draw in the slope at a particular point by drawing the “tangent” at that point and calculating its slope. This option is available under the “Special Menu”, it is called “Differentiate”. *Differentiation is the process involved in finding the Gradient Function of a given function.*

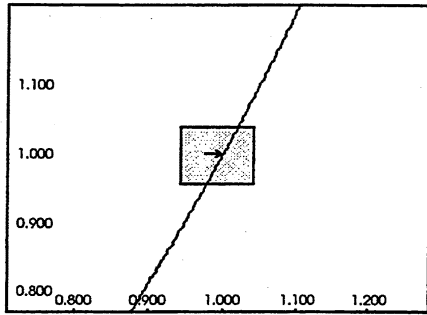
2. Using the Sharp EL-9600

Using the Sharp EL-9600 plot the graph of $D = t^2$. (Using Y -axis for D and the X -axis for t) (ie. Graph $Y1 = x^2$). Press $\backslash x(Y1) \backslash x(X/Q/T/N) \backslash x(X^2) \backslash x(\text{ENTER})$
 Press $\backslash x(\text{GRAPH})$ to see the graph. Using the TRACE option, the coordinate values on the graph can be identified using the pointer, investigate the answers to the preceding questions with reference to the graph.. Also since we are investigating the position of the vehicle after it leaves its starting position we need only consider $t > 0$.
 (Use the $\backslash x(\text{WINDOW})$ button to adjust the Domain if you need to.)

Speed:

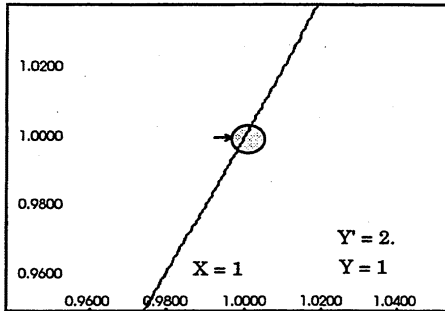


Instantaneous Speed; We can see that the Slope or Gradient of the curve(function) can be used to determine the speed, however there is a problem here in that we are dealing with a curve rather than a straight line. the instantaneous speed is a little more difficult to find but not impossible. Finding the speed when $t = 1$ hr requires an examination of the graph in the “neighbourhood” of $t = 1$.



By using the “Zoom” feature on the calculator we can “zoom in” \x(ZOOM) \x(3) on the point (1, 1) and examine the slope to establish the speed of the vehicle at this point.

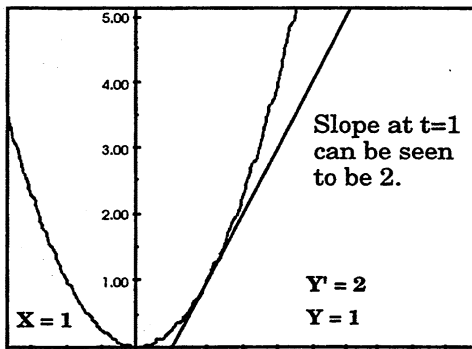
We notice that as the zoom area gets smaller the graph appears to get straighter.



(As we zoom in closer and closer we could draw a line with slope exactly 2, just touching the graph at (1, 1). (ie A tangent to the curve at $t=1$). The slope of this tangent would be the same as the slope of the curve at that point. To do this on the calculator use;

\x(2nd F), \x(DRAW), \x(5 T_LINE), \x(ENTER)

move the cursor to the point (1,1) and press \x(ENTER).



Zooming Out \x(ZOOM) \x(4) reveals that this line is a tangent to the curve and its gradient is the gradient of the curve at (1, 1). The Sharp EL-9600 allows us to calculate this value; \x(2nd F), \x(D Y'), \x(1 ON), \x(ENTER), \x(GRAPH), \x(TRACE)

Y' appears on the screen showing the value of the slope of the tangent at the designated point to be equal to 2.(ie. at time $t = 1$ hour, the vehicle is travelling at 2 km/hr.)

To assist us with the process of calculating instantaneous speed, the calculator will draw in the slope at a particular point by drawing the “tangent” at that point and calculating its slope.

We can DRAW a TANGENT at a point using the Calculator

Press \x(GRAPH), \x(2nd F), \x(STAT(Draw)), \x(5 T_Line), \x(ENTER)

Use the Calculator to read values of the graph for Y' on the curve $Y1 = X^2$ and complete the following table. Make sure that \x(TRACE) is on and that Y' is showing.

Use \x(2nd F), \x(ZOOM), \x(D_Y'), \x(1) \x(ENTER) to turn Y' on.

X	-4	-3	-2	-1	0	1	2	3	4
Y'									

What FUNCTION is represented by these values

\x($Y' =$) Use the Calculator to check.

Graph $Y1 = 2X$

Use \x(ZOOM) \x(7) to set increments at 0.1 and then switch to \x(TABLE) and check that the values agree with those in the table above.

Conclusion: Therefore we see that if $Y = X^2$ the $Y' = 2X$

Exercise: Plot the Function $Y1 = X^3$

Use the Y' function to find values of the gradient (slope) of this function and fill in the table below.

X	-4	-3	-2	-1	0	1	2	3	4
Y'									

Guess a function rule. What function is represented by these values

(Y' =)

Test this by graphing the function (Y1) (2) (X/Q/T/N) and then checking the values in your table above against the (TABLE) on the calculator.

Write down a general rule for the gradient function of $y = x^n$.

Rule: (Y' =)

Check your rule for $y = x^4$ and $y = x^5$ by graphing the function, selecting values of y' and then using the (TABLE) on the calculator to test your prediction.

Extension: Work out gradient functions for the following and test your predictions

using the calculator. 1. $y = x^2 + x$; 2. $y = 3x^2$; 3. $y = 2x^3 + x^2$

References

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